Eigenfrequencies of Nonuniform Beams

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Summary

Most of the approximate methods (Rayleigh-Ritz, iteration, etc.) for computing eigenfrequencies of structures yield upper bounds. Lower bounds are usually less easy to estimate and often present larger discrepancies from the exact values. There are cases, however, where methods of mass and/or flexibility decomposition, applicable to composite systems, readily give useful results. Stepped beams are a good case in point. Further problems of beams with variable cross sections are considered and formulas are given for the two first eigenfrequencies of symmetric beams.

Introduction

Amongst recent papers devoted to the problem of computing eigenfrequencies of elastic structures, that of Taleb and Suppieger[1] gave a nice application of Cauchy's iteration method to the case of a stepped beam. It is worth noting that such methods, as also energy methods like the Rayleigh-Ritz technique give upper bounds for the first eigenfrequency. On the other hand, this problem lends itself to an easy application of the so-called decomposition method,[2] which yields a lower bound for the same eigenvalue.

In short, this method consists in determining n "partial" systems such that the sum of their influence coefficients $G_i(x, \xi)$ is equal to the influence coefficient of the given system:

$$G(x, \xi) = \sum_{i=1}^{n} G_i(x, \xi) \tag{1}$$

These influence coefficients (or Green functions), which are symmetric functions of the variables x and ξ along the beam, represent the deflections at abscissa x due to unit forces acting at abscissa ξ (see Appendix 1).

Assuming that the first frequencies ω_{i1} of these partial systems are easy to determine, a lower bound for the first frequency ω_1 of the original system is given by

$$\omega_1^{-2} \leqslant \sum_{i=1}^{n} \omega_{i1}^{-2} \tag{2}$$

Before considering more general cases of beams with variable mass and rigidity distributions, it seems interesting to exemplify the method of decomposition in the case of a stepped beam.

Stepped Beam

Fig. 1a represents the considered system; it can be decomposed into two partial systems by assuming infinite rigidity alternatively in the right and the left sections (Figs. 1b and 1c). It is then easy to verify that Eq. (1) is satisfied.

Let us consider the first system (Fig. 1b). The equation of motion is

$$EI_1(\partial^4 y/\partial x^4) = -m_1(\partial^2 y/\partial t^2) \tag{3}$$

where y is the deflection between $x = 0$ and $x = l$, and EI_1 and m_1 are the bending modulus and the mass per unit length in this section, respectively. Writing

$$y(x, t) = Y(x)e^{j\omega t} \tag{4}$$

Eq. (3) becomes

$$EI_1 d^4Y/dx^4 = m_1\omega^2 Y \tag{5}$$

with the following boundary conditions when both ends are hinged:

$$Y(l_1) = a \tag{6}$$

$$Y'(l_1) = -a/l_2 \tag{7}$$

$$EI_1[l_2Y'''(l_1) + Y''(l_1)] = -m_2\omega^2 a l_2^2/3 \tag{8}$$

Conditions (6) and (7) are purely geometric and allow elimination of the arbitrary amplitude a at the step cross section. Condition (8) is obtained by writing that the moment of all the forces acting on the rigid section, with respect to its right hinge, vanishes; at the first member appear the shear force

$$Q = -EI_1Y''' \tag{9}$$

and the bending moment

$$M = -EI_1Y'' \tag{10}$$

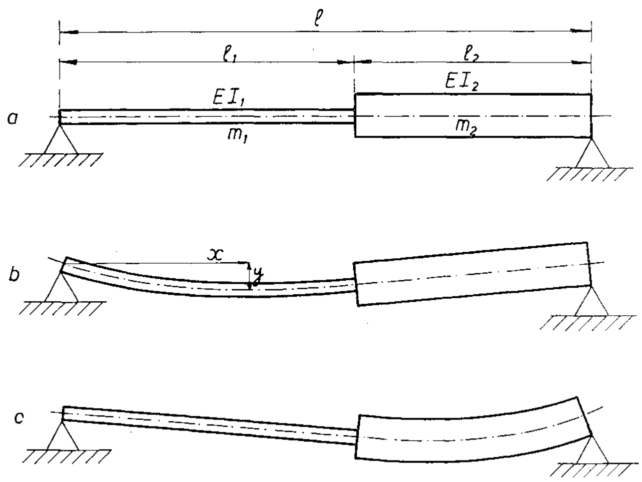

Fig. 1. Splitting the flexural rigidity of a stepped beam (a) in two parallel systems (b) and (c).

Received by IAS June 25, 1962.

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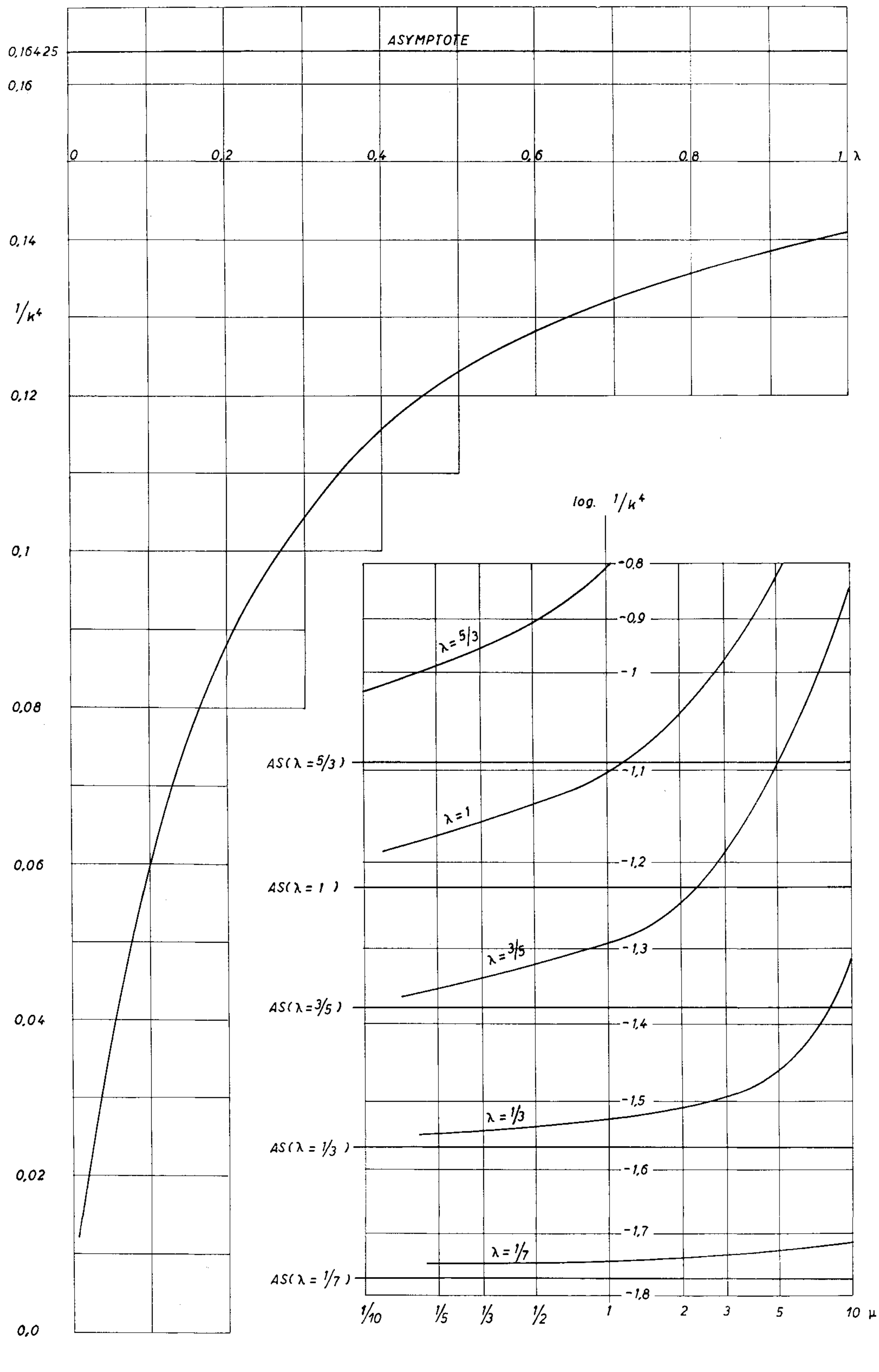

FIG. 2. Curves of $k^{-4} = EI_1/m_1\,\omega^2 l_1^4$ as a function of $\mu = m_2/m_1$ for different values of $\lambda = l_2/l_1$. Left and above: asymptotic value of k^{-4} (for $\mu = 0$) as a function of λ.

whereas at the second member stand the inertia forces of the same section.

This is an easy uniform beam problem, and by introducing the nondimensional parameters

$$\left.\begin{aligned} k &= l_1(\omega^2 m_1/EI_1)^{1/4} \\ \lambda &= l_2/l_1, \quad \mu = m_2/m_1 \end{aligned}\right\} \tag{11}$$

the frequency equations take the form

$$\mu(k\lambda)^3[\sin(k)\cosh(k) - \cos(k)\sinh(k)]/6 - (k\lambda)^2\cos(k)\cosh(k) - k\lambda[\sin(k)\cosh(k) + \cos(k)\sinh(k)] - \sin(k)\sinh(k) = 0 \tag{12}$$

For a given system, the ratios λ and μ are determined and the first frequency is to be computed from the lowest root k_1 of this equation.

There are only two independent parameters, λ and

μ, and therefore it is easy to plot a diagram of k as a function of μ, as a set of curves for different values of λ. As, for the second partial system, μ and λ assume inverse values, a logarithmic scale is provided for this parameter; furthermore, Eq. (2) involves the inverses of ω^2; thus, instead of k, it is k^{-4} which is taken as the dependent variable in Fig. 2. This figure can now be used for any imaginable type of stepped beam (with only one step).

Let us call λ_1 and μ_1 the values of λ and μ for the first partial system (Fig. 1b); then Fig. 2 gives $k_{11}{}^{-4}$, which in turn yields

$$\omega_{11}{}^{-2} = (m_1 l_1{}^4/EI_1)k_{11}{}^{-4} \tag{13}$$

Now the second partial system of Fig. 1c has for λ and μ the values $\lambda_2 = 1/\lambda_1$ and $\mu_2 = 1/\mu_1$, and from Fig. 2 one finds the value of $k_{21}{}^{-4}$, whence

$$\omega_{21}{}^{-2} = (m_2 l_2{}^4/EI_2)k_{21}{}^{-4} \tag{14}$$

and the lower bound for the original system, following Eq. (2), is

$$\omega_1{}^{-2} \leqslant \omega_{11}{}^{-2} + \omega_{21}{}^{-2} \tag{15}$$

In Ref. 1 the particular case in which $\lambda_1 = 1$, $\mu_1 = 2$, and $\alpha_1 = 4$ is solved. Applying Cauchy's iteration method, the authors find

$$\omega_1 = (10.8/l^2)(EI_1/m_1)^{1/2}$$

which effectively is an upper bound for the exact value, which happens to be

$$\omega_1 = (10.5/l^2)(EI_1/m_1)^{1/2} \tag{16}$$

Applying our method, we obtain for $\lambda_1 = 1$, $\mu_1 = 2$:

$$k_{11}{}^{-4} = 0.11326$$

and for $\lambda_2 = 1$, $\mu_2 = 0.5$:

$$k_{21}{}^{-4} = 0.07237$$

Now from Eq. (15) we find

$$\omega_1{}^{-2} \leqslant (m_1 l_1{}^4/EI_1)[k_{11}{}^{-4} + (\mu_1\lambda_1/\alpha_1)k_{21}{}^{-4}]$$

or

$$\omega_1{}^{-2} \leqslant 0.14944(m_1 l_1{}^4/EI_1) = 0.00934(m_1 l^4/EI_1)$$

whence the lower bound

$$\omega_1 \geqslant (10.35/l^2)(EI_1/m_1)^{1/2}$$

which is seen to be quite close to the exact value.

The shapes of the curves in Fig. 2 suggest that some further simplification could be introduced by using their asymptotes (*as*). This is obtained by further decomposition of each partial system following Dunkerley's formula.[3] This means here that in turn one portion of the beam alone will retain its mass.

Thus, in each partial system there will be two subsystems obtained by assuming for the first $m_2 = 0$ (or $\mu = 0$), and for the second $m_1 = 0$ (or $\mu = \infty$). Instead of Eq. (12) of partial system b, for which $EI_2 = \infty$, we now have the two equations

$$(k'\lambda)^2 \cos(k')\cosh(k') - k'\lambda[\sin(k')\cosh(k') + \cos(k')\sinh(k')] - \sin(k')\sinh(k') = 0 \tag{17}$$

$$\omega''^{-2} = (m_1 l_1{}^4/EI_1)k''^{-4} \quad \text{with } k''^{-4} = \mu\lambda^2/9(1+\lambda)^2 \tag{18}$$

Eq. (17) gives the horizontal asymptotes $k' = k_\infty$ for different values of λ; the curve $k'^{-4} = f(\lambda)$ is also given in Fig. 2.

Eq. (18) for the frequency of the second subsystem is readily obtained by solving Eq. (3) with $m_1 = 0$, and conditions (6), (7), and (8).

Thus, in the example considered above with $\lambda_1 = l_2/l_1 = 1$, $\mu_1 = m_2/m_1 = 2$, and $\alpha = EI_2/EI_1 = 4$, a lower bound for the frequency of the first partial system is given by

$$k_{11}{}^{-4} \leqslant k_{11}'^{-4} + k_{11}''^{-4}$$

From Fig. 2 one reads $k_{11}'^{-4} = 0.058705$, while Eq. (18) yields $k_{11}''^{-4} = 1/18 = 0.055556$; and effectively

$$k_{11}{}^{-4} = 0.11326 < 0.11426$$

Similarly, in the second partial system we find

$$k_{21}{}^{-4} = 0.07237 < 0.058705 + 0.013888 = 0.07259$$

Thus if, instead of using Fig. 2 with two partial systems as above, we had taken four subpartial systems as given by the following scheme:

Subpartial systems	EI_1	EI_2	m_1	m_2	k^{-4}
1′	EI_1	∞	m_1	0	0.058705
1″	EI_1	∞	0	m_2	0.055556
2′	∞	EI_2	0	m_2	0.058705
2″	∞	EI_2	m_1	0	0.013888

we should have found, instead of the approximation (18), a somewhat less accurate lower bound:

$$\omega_1{}^{-2} < (m_1 l_1{}^4/EI_1)[k_{11}'^{-4} + k_{11}''^{-4} + (\mu_1\lambda_1/\alpha_1)(k_{21}'^{-4} + k_{21}''^{-4})]$$

and thus

$$\omega_1 \geqslant (10.31/l^2)(EI_1/m_1)^{1/2}$$

which may be compared with Eq. (16).

These methods of decomposition can be applied in cases where the end conditions are different; but the simplicity of the decomposition depends on the fact that the considered system is statically determinate.

Another statically determinate case is the stepped cantilever (Fig. 3). There is here a close similarity with the problem of the natural vibration of a T-pole in its plane, a case which has been solved by the decomposition method in Ref. 2.

The first partial system (I) assumes that only the clamped section of the beam remains elastic, the rest being rigid, whereas in the second partial system (II) it is only this clamped section which is rigid. Computation of the lower bound for ω_1 is further simplified by

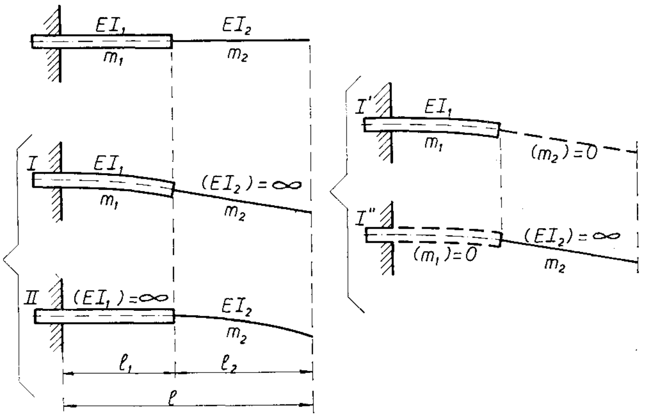

FIG. 3. Splitting the flexural rigidity of a stepped cantilever in two partial systems, I and II, with further splitting of the mass of partial system I in two subsystems, I′ and I″.

subdividing the first partial system in two subsystems, this time following Dunkerley's procedure: the rigidity distribution is the same, but m_2 is assumed zero in subsystem I', and m_1 assumed zero in subsystem I''.

If ω_{11}', and ω_{11}'' are the corresponding first frequencies, Dunkerley's formula[3] gives the approximation

$$\omega_{11}^{-2} < \omega_{11}'^{-2} + \omega_{11}''^{-2} \tag{19}$$

which allows retaining the inequality (2).

These frequencies are easy to compute; the first system is a simple uniform beam of length l_1, thus

$$\omega_{11}'^2 = 0.127(EI_1/m_1)(\pi/l_1)^4 \tag{20}$$

while the second consists of a rigid beam of mass m_2l_2, elastically suspended at the end of a cantilever devoid of mass.

In this case one finds (see Appendix 2)

$$\omega_{11}''^2 = (12EI_1\lambda/m_2l_2^4)[2 + 3\lambda + 2\lambda^2 - (1 + \lambda)(1 + \lambda + \lambda^2)^{1/2}] \tag{21}$$

where $\lambda = l_2/l_1$. As for partial system II, which simply considers a uniform cantilever of length l_2,

$$\omega_{21}^2 = 0.127(EI_2/m_2)(\pi/l_2)^4 \tag{22}$$

Substituting Eqs. (20) and (21) in Eq. (19), and Eqs. (19) and (22) in Eq. (15), one obtains a lower bound for ω_1

$$\omega_1^2 \geqslant 0.127(EI_1/m_1)(\pi/l_1)^4/\{1 + \mu\lambda^4/\alpha + 1.03\mu\lambda^3[2 + 3\lambda + 2\lambda^2 - (1 + \lambda)(1 + \lambda + \lambda^2)^{1/2}]^{-1}\} \tag{23}$$

where $\mu = m_2/m_1$, $\alpha = EI_2/EI_1$, and $0.127\pi^4/12 = 1.03$.

Beams With Continuously Varying Mass and Rigidity Distributions

The method of decomposition can be applied to multistepped beams as represented by Fig. 4a, by considering partial systems where only successive sections would retain their elasticity. For more than two sections, the number of independent parameters λ, μ, etc. becomes prohibitive as far as their diagramming is concerned, following the model of Fig. 2 for the one-stepped beam. Nevertheless, particular cases should be easy to solve individually without undue labor. However, this suggests a more general way of applying the decomposition principle for any distribution of mass and rigidity.

An obvious limiting case of Fig. 4a consists in forming an infinite number of partial systems for which only a portion Δx of the beam would remain elastic (Fig. 4b).

In this rather crude way of simplifying the partial systems, the square of their natural periods of vibration will be proportional to Δx and can be determined from the equilibrium of the two parts separated by the element Δx. Taking the moments with respect to the end points, one finds

$$\frac{\omega_x{}^2 a}{x} \int_0^x m(\xi)\xi^2 d\xi = -Qx + M \tag{24}$$

$$\frac{\omega_x{}^2 a}{(l - x)} \int_0^l m(\xi)(l - \xi)^2 d\xi = Q\,(l - x) + M \tag{25}$$

which allows elimination of the shear component Q. There is, further, a relation between the amplitude a and the bending moment M, for

$$M = EI/\rho \tag{26}$$

where $1/\rho$ is the curvature of the flexible element Δx. Now

$$\Delta x/\rho = a/x + a/(l - x) \tag{27}$$

and thus

$$M = (aEI/\Delta x)[1/x + 1/(l - x)] \tag{28}$$

Substituting Eq. (28) in Eqs. (24) and (25) and eliminating Q gives

$$\omega_x{}^{-2} = (\Delta x/EI)[x^{-1} + (l - x)^{-1}]^{-2} \times \left[x^{-2}\int_0^x m\xi^2 d\xi + (l - x)^{-2}\int_x^l m(l - \xi)^2 d\xi\right] \tag{29}$$

Superposition now gives, as a lower bound for $\omega_1{}^2$,

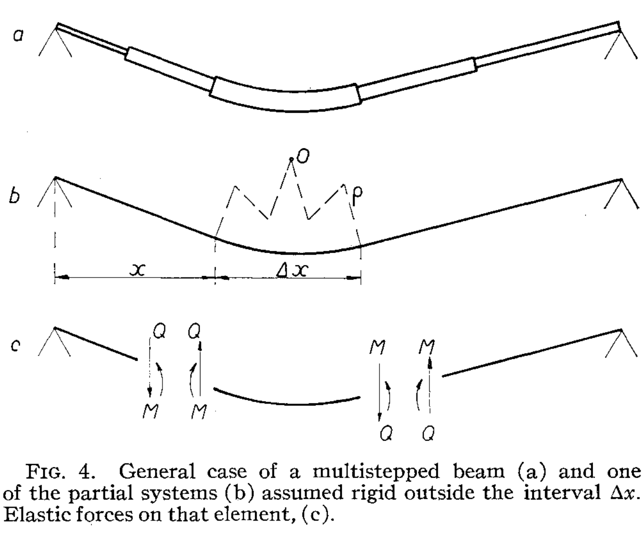

FIG. 4. General case of a multistepped beam (a) and one of the partial systems (b) assumed rigid outside the interval Δx. Elastic forces on that element, (c).

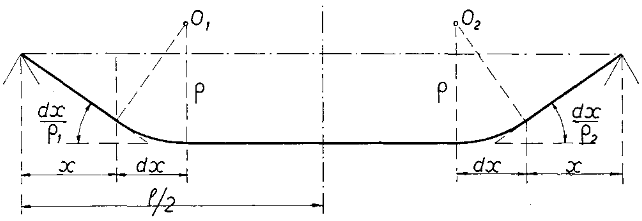

FIG. 5. Case of a symmetric beam with variable cross section. Symmetric partial system where the beam is assumed rigid outside the intervals dx.

$$\omega_1^{-2} \leqslant \int_0^l (dx/EIl^2)\left[(l-x)^2\int_0^x m\xi^2 d\xi + x^2\int_x^l m(l-\xi)^2 d\xi\right] \quad (30)$$

In fact the different deflections assumed by Fig. 4b for $0 < x < l$ are too different from the actual one, which is a half sine wave, to permit any hope of a good approximation. In the case of a uniform beam (EI and m constant), Eq. (30) gives the inequality

$$(EI/m)(\pi/l)^4 = \omega_1 > (90\ EI/ml^4) \quad (31)$$

which is correct but too pronounced to be considered as an approximation, for $\pi^4 = 97.409$.

A significant improvement is obtained by taking partial systems with *two* elastic hinges located symmetrically on the beam, but in the general case it leads to rather cumbersome formulas.

Specializing to the more important case where rigidity and mass are symmetrically distributed

$$EI(x) = EI(l-x), \quad m(x) = m(l-x)$$

the equations of equilibrium needed can be written for a half-span; thus besides Eq. (24), we write

$$\omega_{x1}^2 a\int_x^{l/2} m d\xi = T \quad (32)$$

for the middle section, whose motion is only translatory (Fig. 5).

Instead of Eqs. (27), we now have

$$M = EI/\rho = EIa/x\Delta x \quad (33)$$

and eliminating T between Eqs. (24) and (32), we find

$$\omega_{x1}^{-2} = (\Delta x/EI)\left[\int_0^x m\xi^2 d\xi + x^2\int_x^{l/2} m d\xi\right] \quad (34)$$

and by superposition

$$\omega_1^{-2} \leqslant \int_0^{l/2} (dx/EI)\left[\int_0^x m\xi^2 d\xi + x^2\int_x^{l/2} m d\xi\right] \quad (35)$$

If m were constant:

$$\omega_1^{-2} \leqslant m\int_0^{l/2} (x^2/dx/EI)(l/2 - 2x/3) \quad (36)$$

and if further EI were also constant, Eq. (36) would give

$$\omega_1^2 \geqslant 96El/ml^4 \quad (37)$$

This is a far better approximation ($\pi^4 = 97.409 > 96$); therefore, if EI and m are no longer constant but are given functions of x, the integration of Eq. (35) can be considered as giving a useful lower bound for the first frequency. On the other hand, excellent upper bounds can always be obtained by applying energy or iteration methods.

An approximation can also be provided quite easily for the second frequency by noting that [see, e.g., Ref. 3, p. 95, Eq. (7.13)]

$$\int_0^l G(x, x)m(x)dx = \sum_{k=1}^{\infty} \omega_k^{-2} \quad (38)$$

where ω_k is the eigenpulsation of kth order.

As the same relation exists for each of the n partial systems for any decomposition of type (1), one deduces

$$\sum_{k=1}^{\infty} \omega_k^{-2} = \sum_{i=1}^{n}\sum_{k=1}^{\infty} \omega_{1k}^{-2} \quad (39)$$

When summing up for k, it must be understood that if the number of degrees of freedom of the considered system is finite, the summation is only extended to the corresponding number of eigenfrequencies.

For example, the partial systems of Fig. 4b have only one degree of freedom and thus only one eigenfrequency, when Δx is assumed to be vanishingly small. When applied to the case of a beam on two hinges, the value of Eq. (39) (where in the second member the summation with respect to k contains only one term for each partial system i) is directly given by the second member of Eq. (30). Thus Eqs. (30) and (39) are summarized by

$$\omega_1^{-2} < \sum_{k=1}^{\infty} \omega_k^{-2} = \int_0^l (dx/EIl^2)\left[(l-x)^2 \times \int_0^x m\xi^2 d\xi + x^2\int_x^l m(l-\xi)^2 d\xi\right] \quad (40)$$

Further, combining Eqs. (2) and (33), we have

$$\sum_{k=2}^{\infty} \omega_k^{-2} > \sum_{i=1}^{n}\sum_{k=2}^{\infty} \omega_{ik}^{-2} \quad (41)$$

A decomposition with two elastic hinges of the type of Fig. 5 gives two eigenfrequencies to each partial system. When the beam is assumed symmetric, Eq. (40) reduces to

$$\sum_{k=1}^{\infty} \omega_k^{-2} = \sum_{i=1}^{n}\sum_{k=1}^{\infty} \omega_{ik}^{-2} = 4\int_0^{l/2} (dx/EIl^2) \times [l(l/2 - x)\int_0^x m\xi^2 d\xi + x^2\int_0^{l/2} m\xi^2 d\xi] \quad (42)$$

and Eq. (35), whose second member gives $\sum_{i=1}^{n} \omega_{i1}^{-2}$, can be used to write

$$\sum_{k=2}^{\infty} \omega_k^{-2} > \sum_{i=1}^{n}\left(\sum_{k=1}^{\infty} \omega_{ik}^{-2} - \omega_{i1}^{-2}\right) > \int_0^{l/2} (dx/EIl^2) \times \left[(l-2x)^2\int_0^x m\xi^2 d\xi + x^2\int_x^{l/2} m(l-2\xi)^2 d\xi\right] \quad (43)$$

If higher harmonics than the second can be neglected in the first member, one could write as an approximation

$$\omega_2^{-2} \cong \int_0^{l/2} (dx/EIl^2)[\,.\,.\,.\,.\,.\,] \tag{44}$$

the quantity between the brackets being the same as in Eq. (43).

In the case where EI and m are constant, this gives

$$\sum_{k=2}^{\infty} \omega_k^{-2} > (ml^4/EI)(1/90 - 1/96) = ml^4/1440EI \tag{45}$$

This shows that Eq. (44) may give an acceptable first approximation, for in that case one has, according to Eq. (44)

$$\omega_2 = (37.947/l^2)\sqrt{EI/m} \tag{46}$$

which, compared with the exact value

$$\omega_2 = (4\pi^2/l^2)\sqrt{EI/m} \tag{47}$$

where $4\pi^2 = 39.478$, shows an error of 4 percent.

As a last example of how a "continuous" decomposition of influence coefficients yields a lower bound for the first frequency, let us again consider a cantilever, this time with an arbitrarily varying mass m and bending modulus EI.

It is seen that introducing an elastic hinge between assumedly rigid portions of the beam gives modes of vibration for the different partial systems which are not too different from the actual first mode (Fig. 6).

Each partial system, defined by the abscissa x of the hinge, has only one degree of freedom, and the corresponding frequency is given by

$$\omega_x^2\theta \int_x^l m(\xi),(\xi - x)^2 d\xi = M = EI/\rho = EI\theta/dx \tag{48}$$

which allows elimination of the arbitrary angular amplitude θ (Fig. 6) and yields

$$\omega_x^{-2} = (dx/EI) \int_x^l m(\xi)(\xi - x)^2 d\xi \tag{49}$$

Integrating as indicated by the sum of Eq. (2), we get the lower bound:

$$\omega_1^{-2} \leq \int_0^l \omega_x^{-2} = \int_0^l (dx/EI) \int_x^l m(\xi)(\xi - x)^2 d\xi \tag{50}$$

If the distributed mass m is constant,

$$\omega_1^{-2} \leq \int_0^l dx\, m(l - x)^3/3EI \tag{51}$$

and if EI is also constant

$$\omega_1^2 \geqslant 12EI/ml^4 \tag{52}$$

which is not a bad lower bound, for the actual value is

$$\omega_1^2 = 12{,}39\ EI/ml^4 \tag{53}$$

Appendix 1

Using the influence coefficient $G(x, \xi)$, the equation for a vibrating beam can be put in the integral form:

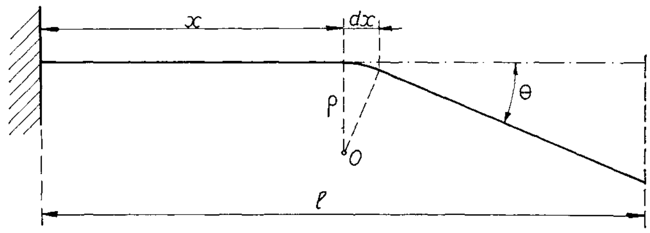

Fig. 6. Case of a cantilever with variable cross section. Partial system where elasticity is retained only in the interval dx.

$$Y(x) = \omega^2 \int_0^l G(x, \xi)m(\xi)Y(\xi)d\xi \tag{A1}$$

or in the more classical mold:

$$\phi(x) = \lambda \int_0^l K(x, \xi)\phi(\xi)d\xi \tag{A2}$$

where $\phi(x) = Y(x)\sqrt{m(x)}$, $\lambda = \omega^2$, and

$$K(x, \xi) = G(x, \xi)\sqrt{m(x)m(\xi)} \tag{A3}$$

which is the "kernel" of the integral equation.

Now the recurrent maximum kinetic energy of the vibrating beam is given by

$$2K = \omega^2 \int_0^l m(x)Y^2(x)dx = \lambda \int_0^l \phi^2(x)dx \tag{A4}$$

whereas its double maximum potential energy is

$$2U = \omega^4 \int_0^l \int_0^l G(x, \xi)m(x)m(\xi)Y(x)Y(\xi)dxd\xi =$$

$$\lambda^2 \int_0^l \int_0^l K(x, \xi)\phi(x)\phi(\xi)dxd\xi \tag{A5}$$

or, using the bending modulus EI,

$$2U = \int_0^l EIY''^2dx + F(Y(0), Y(1), \ldots) \tag{A6}$$

where F is only a function of Y and its derivatives at the ends of the beam (case of elastic hinges, etc.). Putting $K = U$ yields

$$\omega^{-2} = \int_0^l \int_0^l G(x, \xi)m(x)m(\xi)Y(x)Y(\xi)dxd\xi \Big/ \int_0^l m(x)Y^2(x)dx \tag{A7}$$

$$= \int^l \int^l K(x, \xi)\phi(x)\phi(\xi)dxd\xi \Big/ \int_0^l \phi^2(x)dx \tag{A8}$$

and also

$$\omega^{-2} = \int_0 m(x)Y^2(x)dx \Big/ \left[\int_0^l EIY''^2dx + F(Y(0), Y(1), \ldots)\right] \tag{A9}$$

Eq. (A7) shows that a decomposition Eq. (1) applied to $G(x, \xi)$ gives Eq. (2) according to Rayleigh's principle, for the actual deflection Y does not necessarily coincide with the deflection of each partial system.

Eq. (A8) suggests that decomposition (1) could be generalized in the form:

$$K(x, \xi) = \sum_{i=1}^{n} K_i(x, \xi) \tag{A10}$$

Eq. (A9) shows that result (2) is also obtained by Dunkerley's mass decomposition:

$$m(x) = \sum_{i=1}^{n} m_i(x) \tag{A11}$$

Appendix 2

The two natural frequencies of an oscillator consisting of a mass M attached at the end of a cantilever of length l_1 whose own mass is neglected, are obtained by solving the equation

$$EI_1 Y''' = -\omega^2 M(Y_1 + Y_1'b) \tag{B1}$$

where $Y_1 = Y(l_1)$ and $Y_1' = Y'(l_1)$, the center of mass M being at distance $b = l_2/2$ from the end.

Integration gives

$$EI_1 Y'' = -\omega^2 M(Y_1 + Y_1'b)x + C \tag{B2}$$

where C is determined by the end condition involving the inertia moment J of the mass M around its own center,

$$EI_1 Y_1'' = \omega^2[JY_1' + bM(Y_1 + Y_1'b)] \tag{B3}$$

Thus,

$$C = \omega^2[JY_1' + M(Y_1 + Y_1'(b)(l_1 + b)] \tag{B4}$$

Further integration, with the conditions $Y(0) = Y'(0) = 0$, yields

$$EI_1 Y' = -\omega^2 M(Y_1 + Y_1'b)x^2/2 + Cx \tag{B5}$$

$$EI_1 Y = -\omega^2 M(Y_1 + Y_1'b)x^3/6 + Cx^2/2 \tag{B6}$$

Taking $x = l_1$ in these two last equations and eliminating Y_1 and Y_1' gives the frequency equation

$$\rho^4\gamma - 2\rho^2(\beta + \gamma) + 1 = 0 \tag{B7}$$

where

$$\rho^2 = \omega^2 M l_1^3/6EI_1$$

$$\beta = 1 + 3b/l_1 + 3(b/l_1)^2$$

$$\gamma = 3J/Ml_1^2$$

Whence the frequencies are given by

$$\rho^2 = \beta\gamma^{-1} + 1 \pm \sqrt{(\beta\gamma^{-1} + 1)^2 - \gamma^{-1}} \tag{B8}$$

If M is a beam of length l_2 and uniformly distributed mass m_2,

$$b = l_2/2; \ \gamma = b^2/l_1^2$$

$$\beta = 1 + 3l_2/2l_1 + 3l_2^2/4l_1^2$$

and Eq. (B8) gives

$$\omega^2 = (12EI_1/m_2 l_2^3 l_1^3)[2l_1^2 + 3l_1 l_2 + 2l_2^2 \pm (l_1 + l_2)\sqrt{l_1^2 + l_1 l_2 + l_2^2}] \tag{B9}$$

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